The distribution of second degrees in the Buckley–Osthus random graph model

Andrey Kupavskii[*], Liudmila Ostroumova[†], Dmitriy Shabanov[‡], Prasad Tetali[§]

Abstract

In this paper we consider a well-known generalization of the Barabási and Albert preferential attachment model – the Buckley-Osthus model. Buckley and Osthus proved that in this model the degree sequence has a power law distribution. As a natural (and arguably more interesting) next step, we study the second degrees of vertices. Roughly speaking, the second degree of a vertex is the number of vertices at distance two from this vertex. The distribution of second degrees is of interest because it is a good approximation of PageRank, where the importance of a vertex is measured by taking into account the popularity of its neighbors.

We prove that the second degrees also obey a power law. More precisely, we estimate the expectation of the number of vertices with the second degree greater than or equal to k and prove the concentration of this random variable around its expectation using the now-famous Talagrand's concentration inequality over product spaces. As far as we know this is the only application of Talagrand's inequality to random web graphs, where the (preferential attachment) edges are not defined over a product distribution, making the application nontrivial, and requiring certain novelty.

Keywords: random graphs, preferential attachment, power law distribution, second degrees.

1 Introduction

In this paper we consider some properties of random graphs. The standard random graph model $G(n, m)$ was introduced by Erdős and Rényi in [12]. In this model we randomly choose one graph from all graphs with n vertices and m edges. The similar model $G(n, p)$ was suggested

[*]Yandex, Department of theoretical and practical research; Department of Discrete Mathematics, Moscow Institute of Physics and Technology; Moscow State University, Faculty of Mechanics and Mathematics, Department of Number Theory.

[†]Yandex, Department of theoretical and practical research; Moscow State University, Mechanics and Mathematics Faculty, Department of Mathematical Statistics and Random Processes.

[‡]Yandex, Department of theoretical and practical research; Department of Discrete Mathematics, Moscow Institute of Physics and Technology; Moscow State University, Faculty of Mechanics and Mathematics, Department of Probability Theory.

[§]School of Mathematics and School of Computer Science, Georgia Institute of Technology, Atlanta, GA, USA

by Gilbert in [14]: we have an n-vertex set and we join each pair of vertices independently with probability $0 < p < 1$. Many papers deal with the classical models. Fundamental results about these models can be found in [5], [13], [16].

Recently there has been an increasing interest in modeling complex real-world networks. It is well understood that real structures differ from standard random graphs. Many models of real-world networks and main results can be found in [6]. For example, a basic characteristic of random graphs is their degree sequence. In many real-world structures the degree sequence obeys a power law distribution. However, standard random graph models do not have this property.

In 1999, Barabási and Albert [3] suggested the so-called preferential attachment model that has a desired degree distribution. Later Bollobás and Riordan [7] gave a more precise definition of this model. In this model the probability that a new vertex is connected to some previous vertex v is proportional to the degree of v. Bollobás and Riordan also proved that the degree sequence has a power law distribution with exponent equal to -3.

Naturally one would not expect that this constant will suit all (or even most) of the real networks. In order to make the model more flexible, two groups of authors (see [10] and [11]) proposed to add one more parameter — an "initial attractiveness" of a node which is a positive constant that does not depend on the degree. In [9], Buckley and Osthus gave an explicit construction of that model.

Many papers deal with different variations of preferential attachment. We mention here the paper by Rudas, Tóth and Valko (see [18]). The authors consider quite a generic model of a random tree and prove some interesting results concerning a neighborhood structure of a random vertex. Also one can find a neighborhood analysis in preferential attachment models in the preprint [4] on the weak graph limit.

This paper deals with the Buckley–Osthus model, which we now describe. Let n be a number of vertices in our graph, $m \in \mathbb{N}$ and $a \in \mathbb{R}_+$ be fixed parameters.

We begin with the case $m = 1$. We inductively construct a random graph $H^n_{a,1}$. The graph $H^1_{a,1}$ consists of one vertex and one loop (we can also start with $H^0_{a,1}$, which is the empty graph). Assume that we have already constructed the graph $H^{t-1}_{a,1}$. At the next step we add one vertex t and one edge between vertices t and i, where i is chosen randomly with

$$\mathsf{P}(i = s) = \begin{cases} \frac{d_{H^{t-1}_{a,1}}(s) - 1 + a}{(a+1)t - 1} & \text{if } 1 \leqslant s \leqslant t-1, \\ \frac{a}{(a+1)t-1} & \text{if } s = t. \end{cases}$$

Here $d_{H^t_{a,1}}(s)$ is the degree of the vertex s in $H^t_{a,1}$. We will also use the notation $d(s) := d_{H^n_{a,1}}(s)$.

To construct $H^n_{a,m}$ with $m > 1$ we start from $H^{mn}_{a,1}$. Then we identify the vertices $1, \dots, m$ to form the first vertex; we identify the vertices $m+1, \dots, 2m$ to form the second vertex, etc. As for the edges, if the edge e connects vertices $im + k$ and $jm + l$, $1 \leqslant k, l \leqslant m$, in the graph $H^{mn}_{a,1}$ then we draw an edge e' between vertices $i+1$ and $j+1$ in $H^n_{a,m}$. Note that we have a one-to-one correspondence between the edges of $H^{mn}_{a,1}$ and $H^n_{a,m}$, so there may be multiple edges (and multiple loops) between vertices in $H^n_{a,m}$. Denote by $\mathfrak{H}^n_{a,m}$ the probability space of constructed graphs.

In [9] Buckley and Osthus proved that the degree sequence of $H^n_{a,m}$ has a power law with exponent $-2 - a$ if a is a natural number. Recently Grechnikov substantially improved this result.

Theorem 1. (Grechnikov, [15]) *Let $a \in \mathbb{R}_+$. If $d = d(n) \geqslant m$ and $\psi(n) \to \infty$ as $n \to \infty$, then*

$$\left| R(d,n) - \frac{\mathrm{B}(d-m+ma, a+2)n}{\mathrm{B}(ma, a+1)} \right| \leqslant \left(\sqrt{d^{-2-a}n} + d^{-1} \right) \psi(n)\,,$$

with probability tending to 1 *as $n \to \infty$. Here $R(d,n)$ is the number of vertices in $H^n_{a,m}$ with degree equal to d and $\mathrm{B}(x,y)$ is the beta function.*

In this paper we consider the so-called second degrees of vertices. Roughly speaking, the second degree of a vertex is the number of vertices at distance two from the vertex. We prove that the number of vertices $Y_n(k)$ with the second degree at least k decreases as k^{-a}, where a is the initial attractiveness. This means that the distribution of second degrees obeys power law. To prove this we calculate the expectation of $Y_n(k)$ and show the concentration of this random variable around its expectation using Talagrand's inequality. The application of this inequality is nontrivial, in particular, we have to redefine the probability space of the Buckley-Osthus graph so that we obtain a product probability space with a product measure. After that we use Talagrand's inequality in its general asymmetric form, which is essential. Verifying the hypothesis of Talagrand's theorem for the present purpose turns out to be delicate, requiring us to introduce additional combinatorial constructions.

This paper is organized as follows. In Section 2 we give the main definitions and formulate the results. In Sections 3 and 4 we prove the theorems stated in Section 2.

2 Definitions and results

2.1 Definitions

In this paper we study the random graph $H^n_{a,1}$. We shall write $ij \in H^n_{a,1}$ if $H^n_{a,1}$ contains the edge ij; we shall write $t \in H^n_{a,1}$ if t is a vertex of $H^n_{a,1}$. Given a vertex $t \in H^n_{a,1}$, the *second degree* of the vertex t is

$$d_2(t) = \#\{ij : i \neq t, j \neq t, it \in H^n_{a,1}, tj \in H^n_{a,1}\}.$$

In other words, the second degree of t is the number of edges adjacent to the neighbors of t except for the edges adjacent to the vertex t. We say that a vertex t is a *k-vertex* if $d_2(t) \geqslant k$.

Let $M^1_n(d)$ be the expectation of the number of vertices with degree d in $H^n_{a,1}$:

$$M^1_n(d) = \mathsf{E}\left(\#\{t \in H^n_{a,1} : d_{H^n_{a,1}}(t) = d\} \right).$$

Let $Y_n(k)$ denote the number of k-vertices in $H^n_{a,1}$.

In this paper we study second degrees of vertices in $H^n_{a,1}$. The main results are stated in Theorems 2 and 5.

We also consider the variable $X_n(k)$ equal to the number of vertices with second degree k in $H^n_{a,1}$. Note that $Y_n(k) = \sum_{i \geqslant k} X_n(i)$.

2.2 Expectation

Theorem 2. *For any $k > 1$ we have*

$$\mathsf{E}Y_n(k) = \frac{(a+1)\Gamma(2a+1)}{\Gamma(a+1)k^a} n \left(1 + O\left(\frac{(\ln k)^{\lceil a+1 \rceil}}{k}\right) + O\left(\frac{k^{1+a}}{n}\right)\right).$$

The easy consequence of Theorem 2 is

Corollary 1. *We have $\mathsf{E}Y_n(k) = \Theta\left(\frac{n}{k^a}\right)$ for $k = O\left(n^{\frac{1}{1+a}}\right)$.*

Using the same technique as in proof of Theorem 2 we can prove the following

Theorem 3. *For any $k \geqslant 1$ we have*

$$\mathsf{E}X_n(k) = \frac{(a+1)\Gamma(2a+1)n}{\Gamma(a)k^{a+1}} \left(1 + O\left(\frac{(\ln k)^{\lceil a+1 \rceil}}{k}\right) + O\left(\frac{k^{1+a}}{n}\right)\right).$$

Again, as a consequence we get

Corollary 2. *We have $\mathsf{E}X_n(k) = \Theta\left(\frac{n}{k^{1+a}}\right)$ for $k = O\left(n^{\frac{1}{1+a}}\right)$.*

We need the following definition. Let $N_n(l,k)$ be the number of vertices in $H^n_{a,1}$ with degree l, with second degree k, and without loops:

$$N_n(l,k) = \#\{t \in H^n_{a,1} : d(t) = l, d_2(t) = k, tt \notin H^n_{a,1}\}.$$

To prove Theorem 2 we need the following auxiliary theorem.

Theorem 4. *In $H^n_{a,1}$ we have*

$$\mathsf{E}N_n(l,k) = c(l,k)\,(n + \theta(n,l,k)),$$

where $|\theta(n,l,k)| < C(l+k)^{1+a}$. The constants $c(l,k)$ are defined as follows:

$$\begin{aligned}
c(l,0) &= c(0,k) = 0,\\
c(1,k) &= c(1,k-1)\frac{a+k-1}{k+3a+1} + c(k)\frac{a+k-1}{k+3a+1}, \quad k > 0,\\
c(l,k) &= c(l,k-1)\frac{al+k-1}{l(1+a)+k+2a} + c(l-1,k)\frac{l-2+a}{l(1+a)+k+2a}, \quad k > 0, l > 1.
\end{aligned}$$

Here $c(k) = \frac{\mathrm{B}(k-1+a,a+2)}{\mathrm{B}(a,a+1)}$.

To prove these theorems we shall use two lemmas. In [15] Grechnikov obtained the following result.

Lemma 1. *Let $k \geqslant 1$ be natural; then*

$$M^1_n(k) = \frac{\mathrm{B}(k-1+a,a+2)n}{\mathrm{B}(a,a+1)} + \tilde{\theta}(n,k),$$

where $|\tilde{\theta}(n,k)| < \tilde{C}/k$.

Denote by $P_n(l,k)$ the number of vertices in $H_{a,1}^n$ with a loop, with degree l, and with second degree k.

Lemma 2. *For any n we have*

$$\mathsf{E}P_n(l,k) \leqslant p(l,k),$$

where

$$\begin{aligned}
p(2,0) &= P, \\
p(l,0) &= p(l-1,0)\frac{l-2+a}{l(1+a)-2-a}, \quad l \geqslant 3, \\
p(l,k) &= p(l,k-1)\frac{al+k-2a-1}{l(1+a)+k-1-a} + p(l-1,k)\frac{l-2+a}{l(1+a)+k-1-a}, \quad l \geqslant 3, k \geqslant 1.
\end{aligned}$$

Here P is some constant. For the other values of l and k we have $p(l,k) = 0$.

2.3 Concentration

Theorem 5. *Let $\delta > 0$ and $k = O\left(n^{\frac{1}{2+a+\delta}}\right)$. Then for some $\epsilon > 0$ we have*

$$\mathsf{P}\left(|Y_n(k) - \mathsf{E}(Y_n(k))| > \left(\mathsf{E}(Y_n(k))\right)^{1-\epsilon}\right) = \bar{o}(1).$$

It is a concentration result which means that the distribution of second degrees does, as the distribution of degrees, obey (asymptotically) a power law.

This theorem is a non-trivial application of Talagrand's inequality (see [19]). Instead of Talagrand's inequality it is possible to apply Azuma's inequality (see [1]), but (as we show later) the result would have been weaker with Azuma's inequality.

We can prove an analogous result for the value $X_n(k)$.

Theorem 6. *Let $\delta > 0$ and $k = O\left(n^{\frac{1}{4+a+\delta}}\right)$. Then for some $\epsilon > 0$ we have*

$$\mathsf{P}\left(|X_n(k) - \mathsf{E}(X_n(k))| > \left(\mathsf{E}(X_n(k))\right)^{1-\epsilon}\right) = \bar{o}(1).$$

If we substitute $a = 1$ in the Buckley–Osthus model then we obtain the Bollobás–Riordan model [8]. The second degrees in this model were considered in [17]. The concentration of second degrees in [17] was proved using Azuma's inequality. This inequality provided the concentration of $X_n(k)$ around its expectation for all $k = O\left(n^{\frac{1}{6+\delta}}\right)$ (with any positive δ). As stated in Theorem 6 Talagrand's inequality gives the stronger result: for Bollobás–Riordan model we obtain the concentration for all $k = O\left(n^{\frac{1}{5+\delta}}\right)$. We obtain this improvement in spite of the fact that the proof of the concentration of $X_n(k)$ in Theorem 6 uses the concentration of $Y_n(k)$ from Theorem 5, so it is not optimal in this sense.

It is possible to generalize Theorem 5 (and also Theorem 6) to the case of arbitrary $m > 1$. The only problem in this case is that we could not prove an analog of Theorem 2 (or Corollary 1) for $m > 1$ since it demands even more calculations. But one would expect that the following conjecture is true.

Conjecture 1. *For any $m > 1$ and $k = O\left(n^{\min\left\{\frac{1}{2+a},\frac{1}{2a}\right\}}\right)$ we have $\mathsf{E}Y_n^m(k) = \Theta\left(\frac{n}{k^a}\right)$, where $Y_n^m(k)$ is the number of k-vertices in $H_{a,m}^n$.*

We can generalize Theorem 5 in the following way.

Theorem 7. *Suppose Conjecture 1 is true. Let $m \in \mathbb{N}$, $\delta > 0$ and $k = O\left(n^{\min\left\{\frac{1}{2+a+\delta},\frac{1}{2a+\delta}\right\}}\right)$. Then for some $\epsilon > 0$ we have*

$$\mathsf{P}\left(|Y_n^m(k) - \mathsf{E}(Y_n^m(k))| > \left(\mathsf{E}(Y_n^m(k))\right)^{1-\epsilon}\right) = \bar{o}(1).$$

In Subsections 3.1 – 3.4 we prove Theorem 5 (using Corollary 1). In Subsection 3.5 we prove Theorem 6 using Corollary 2. In Subsection 3.6 we present the sketch of the proof of Theorem 7. In Section 4 we prove results from Subsection 2.2 (Theorem 2, Theorem 4, and Lemma 2 in Subsections 4.2, 4.1 and 4.3 respectively). Finally, we prove Theorem 3 in Subsection 4.4.

3 Concentration

3.1 Interpretation of the Buckley–Osthus model in terms of independent variables

We consider the following sequence:

$$1, \xi_1, 2, \xi_2, \ldots, n, \xi_n,$$

where $\xi_1, \ldots, \xi_n$ are mutually independent random variables. For every i, we have $\xi_i : \Omega_i \to \{1, \ldots, 2i-1\}$ (here $(\Omega_i, \mathcal{F}_i, \mathsf{P}_i)$ is some probability space) and

$$\mathsf{P}_i(\xi_i = 2j-1) = \frac{a}{(a+1)i-1} \quad \forall j = 1, \ldots, i,$$

$$\mathsf{P}_i(\xi_i = 2j) = \frac{1}{(a+1)i-1} \quad \forall j = 1, \ldots, i-1.$$

We can interpret the sequence in the following way. Each i is a vertex of a graph. Each ξ_i is an endpoint of the edge that goes from the vertex i. If $\xi_i = 2j-1$, then the edge goes to the vertex j. If $\xi_i = 2j$, then we say that the edge from the vertex i goes to the same vertex as the edge from the vertex j. The value of the variable ξ_j can also be even (say $\xi_j = 2j_1$, for some integer j_1), then the edge from the vertex i is again redirected according to the variable ξ_{j_1}. Finally this process stops at some odd value $2v-1$ and we say that ξ_i (as well as ξ_j and ξ_{j_1}) *leads* to the vertex v. We also say that ξ_i leads to ξ_j.

It is not hard to check that the graph model we obtained is exactly the Buckley–Osthus model. Indeed, at each time step i the in-degree of each vertex $j \in \{1, \ldots, i-1\}$ is equal to the number of variables that lead (directly or indirectly) to the vertex j.

Let us give yet another interpretation of the model described above. Consider a vertex v from the obtained graph. We can think of all the variables that lead to v as connected as a rooted tree, with v as the root. Let $X = \{\xi_{i_1}, \ldots, \xi_{i_d}\}$ be the set of variables that lead to v.

We inductively construct the corresponding tree on d vertices $i_1, \ldots, i_d$. First consider those variables $\xi_{i_1^1}, \ldots, \xi_{i_{l_1}^1}$ from X that lead to v directly. The corresponding vertices $i_1^1, \ldots, i_{l_1}^1$ are adjacent to v in the tree. Suppose we choose all the vertices at distance $\leqslant s$ from v and $i_1^s, \ldots, i_{l_s}^s$ are the vertices at distance s from v. Consider the set $\{\xi_{i_1^{s+1}}, \ldots, \xi_{i_{l_{s+1}}^{s+1}}\}$ of variables that lead to some of $\xi_{i_1^s}, \ldots, \xi_{i_{l_s}^s}$. We join each of the vertices $i_1^{s+1}, \ldots, i_{l_{s+1}}^{s+1}$ to the corresponding vertex from $\{i_1^s, \ldots, i_{l_s}^s\}$. We thus obtain the set of vertices at distance $s+1$ from v.

3.2 Decreasing the number of k-vertices

We fix a value $\mathbf{x} = (x_1, \ldots, x_n)$ of the random vector $\xi = (\xi_1, \ldots, \xi_n)$ from the probability space $\Omega = \prod_{i=1}^n \Omega_i$. The quantity $Y_n(k)$ is a function from Ω to $\mathbb{N}$. We discuss the following question. How can the value $Y_n(k) = Y_n(k, \mathbf{x})$ decrease, if we change one coordinate x_i of the vector $\mathbf{x}$? In other words, we want to find $c(i, \mathbf{x}) = \max_{\mathbf{x}'}(Y_n(k, \mathbf{x}) - Y_n(k, \mathbf{x}'))$, where $\mathbf{x}'$ is an arbitrary vector that differs from $\mathbf{x}$ in exactly the ith coordinate.

Lemma 3. *For any $\mathbf{x} = (x_1, \ldots, x_n)$ and $i \in \{1, \ldots, n\}$ we have $c(i, \mathbf{x}) \leqslant 2k + 1$.*

Proof. It is convenient to think about the tree interpretation of the random variables. If we change a value x_i of one random variable ξ_i to some value x_i', then all the variables that lead to x_i are redirected to x_i'. In terms of the tree interpretation, we pick the branch of the tree in which all the edges lead to x_i: if x_i' is odd, then we link the branch to the vertex with the number $(x_i' + 1)/2$; if x_i' is even, then we link the branch to the variable $\xi_{x_i'/2}$.

We want to interpret the change of one coordinate in terms of the graph $H_{a,1}^n$. Suppose x_i leads to a vertex v. Then all the variables that lead to x_i lead to v. If we change x_i to x_i' and x_i' leads to v', then we change the value of all such variables from v to v'. Or, in terms of $H_{a,1}^n$, we take a bundle of edges in the vertex v and move the bundle to the vertex v'. More precisely, if we had a bundle of edges $(i_1, v), \ldots, (i_d, v)$, then after the change we have the edges $(i_1, v'), \ldots, (i_d, v')$. All the rest stays the same.

Now we go on to the proof. We should show that after the change of the ith coordinate, the number of k-vertices we spoil does not exceed $2k+1$. Suppose we moved a bundle of edges $(i_1, v), \ldots, (i_d, v)$. It is easy to see that we could spoil only the k-vertices that have a common edge with v or v itself. Note that we could not spoil the k-vertices in the neighborhood of v'.

We split the set N_v of the vertices incident to v into two parts: $I = \bigcup_{j=1}^d \{i_j\}$ and $N_v \backslash I$. If $|N_v \backslash I| \geqslant k+1$, then after changing the edges from the bundle, all the k-vertices from $N_v \backslash I$ are still k-vertices. Indeed, all the edges in vertex v except for one are 2-incident edges for any neighbor of v, so there are at least k such edges for every vertex from $N_v \backslash I$. Similarly, if $|I| \geqslant k+1$, then no k-vertices among $i_1, \ldots, i_d$ are spoiled except for at most one, since they are all adjacent to the vertex v'. The only case when some of $i_1, \ldots, i_d$ is spoiled is $i_j = v'$ and so we will not count the edges $(i_1, v'), \ldots, (i_d, v')$ in the second degree of i_j.

Finally, the number of k-vertices we spoil does not exceed $\min\{|I|, k\} + \min\{|N_v \backslash I|, k\} + 1 \leqslant 2k + 1$.

$\square$

We now want to estimate the influence of each variable more accurately. Suppose $Y_n(k, \mathbf{x}) = q$. For each k-vertex v_j, $j = 1, \ldots, q$, we consider a subset of coordinates $K_j = K_{v_j}(\mathbf{x}) = \{i_1^j, \ldots, i_{d_j}^j\}$, such that v_j is a k-vertex for any $\mathbf{y}$ that agrees with $\mathbf{x}$ on the coordinates from

K_j. It is worth noting that K_j depends on $\mathbf{x}$, but is not uniquely defined by it. For any choice of the sets $K_1, \ldots, K_q$, we denote their collection by $\mathcal{K} = \mathcal{K}(\mathbf{x})$. Clearly, $Y_n(k, \mathbf{y}) \geqslant q$, for any $\mathbf{y}$ that agrees with $\mathbf{x}$ on all the coordinates from all $K_j \in \mathcal{K}$.

For each coordinate i, we define its multiplicity $C(i, \mathbf{x}, \mathcal{K}) = |\{j : i \in K_j\}|$.

It is easy to see that for any $\mathbf{x}$ and any $\mathcal{K}$ one has $c(i, \mathbf{x}) \leqslant C(i, \mathbf{x}, \mathcal{K})$. So we have

$$c(i, \mathbf{x}) \leqslant \min\{2k + 1, C(i, \mathbf{x}, \mathcal{K})\} =: c_i(\mathbf{x}, \mathcal{K}).$$

We call a collection $\mathcal{K}$ *stable*, if for every k-vertex v_i we construct all the sets K_i according to the following rule: if K_i contains some of the coordinates that lead to a vertex w, then K_i contains all of the coordinates that lead to w.

Now, it should not be surprising that for such set systems we can prove an analog of Lemma 3. Namely, consider a vector $\mathbf{x}$, the corresponding k-vertices $v_j, j = 1, \ldots, q$, and some stable collection $\mathcal{K}$. Let $i \in \{1, \ldots, n\}$ and $\mathcal{K} \setminus \{i\} := \{K_j \setminus \{i\}, j = 1, \ldots, q\}$. Given i there exist at least $q - c_i(\mathbf{x}, K)$ such k-vertices that are k-vertices for any $\mathbf{x}'$ with $x'_s = x_s$ for all $s \in K_j \setminus \{i\}, j = 1, \ldots, q$. To prove this fact one has to follow the proof of Lemma 3 and make sure that the proof works also for this case. The only additional consideration needed is that the number of k-vertices we loose does not exceed the multiplicity $C(j, \mathbf{x}, \mathcal{K})$.

Lemma 4. *Let $\mathcal{K}$ be a stable collection as described above. We have $Y_n(k, \mathbf{x}) - Y_n(k, \mathbf{x}') \leqslant \sum_{j \in J} c_j(\mathbf{x}, \mathcal{K})$ for any vector $\mathbf{x}'$ such that $x'_i = x_i$ for all $i \in \{1, \ldots, n\} \setminus J$.*

Proof. Suppose we change one coordinate j of $\mathbf{x}$ and obtain some vector $\hat{\mathbf{x}}$. Then we consider $d_j := C(j, \mathbf{x}, \mathcal{K})$ k-vertices $w_1, \ldots, w_{d_j}$ such that $j \in K_{w_i}(\mathbf{x})$. We remove j from each of these sets and we check for each $i = 1, \ldots, d_j$ whether the obtained collection guarantees w_i to be a k-vertex or not. If w_i is a k-vertex then we define $K_{w_i}(\hat{\mathbf{x}}) = K_{w_i}(\mathbf{x}) \setminus \{j\}$. If w_i is not a k-vertex then we exclude the set $K_{w_i}(\mathbf{x})$ from $\mathcal{K}$. At the end of this step we obtain a new collection $\hat{\mathcal{K}}$. To prove the lemma we need one consideration. Namely, instead of changing the edges $(i_1, v), \ldots, (i_d, v)$ to $(i_1, v'), \ldots, (i_d, v')$ we can create a new imaginary vertex w and change the edges to $(i_1, w), \ldots, (i_d, w)$. We denote the obtained graph by G_w. We do not count w as a k-vertex even if it has $\geqslant k$ 2-incident edges. It is easy to check that for this graph the collection $\hat{\mathcal{K}}$ is stable.

The number of k-vertices (except for w) in the graph G_w is definitely not bigger than the same number for the graph corresponding to $\hat{\mathbf{x}}$. Moreover, the multiplicity of each coordinate in $\hat{\mathcal{K}}$ is less than or equal to the corresponding multiplicity in $\mathcal{K}$. We also have $Y_n(k, \mathbf{x}) - Y_n(k, G_w) \leqslant c_j(\mathbf{x}, \mathcal{K})$. Similarly, if $\mathbf{x}'$ differs from $\mathbf{x}$ in l coordinates, then the graph corresponding to $\mathbf{x}'$ has at least as many k-vertices as the graph G obtained by forming l imaginary vertices. Moreover, at each step (if we change the coordinate j' and form the corresponding graph G') we spoil at most $\min\{2k+1, C(j', \mathbf{x}, \mathcal{K})\}$ k-vertices and obtain a stable set system.

Consequently, we have $Y_n(k, \mathbf{x}) - Y_n(k, \mathbf{x}') \leqslant Y_n(k, \mathbf{x}) - Y_n(k, G) \leqslant \sum_{j \in J} c_j(\mathbf{x}, \mathcal{K})$.

□

3.3 Construction of a suitable set $\mathcal{K}$

Lemma 5. *Suppose $Y_n(k, \mathbf{x}) = q$ for some vector $\mathbf{x}$ in the corresponding graph $G_{\mathbf{x}}$. Then we can construct a stable set system $\mathcal{K} = \{K_1, \ldots, K_q\}$ such that $\sum_{i=1}^{n} c_i(\mathbf{x}, \mathcal{K}) \leqslant (4k + 5)q$.*

Proof. First consider the set V of vertices with degree at least $k+2$. Put $NV = \{u : u$ is a neighbor of $v \in V\}$. Note that a vertex from V can also belong to NV. Assume that $|NV| = z$. All vertices from NV are k-vertices. Let BV be the set of vertices from NV which do not have an outcoming edge that goes to V. We have $|BV| \leqslant z/(k+1)$ since each vertex has at most one outcoming edge.

We denote by $L_v, L_v \subset \{1, \ldots, n\}$, the set of coordinates that lead to v. We also put $LV = \cup_{v \in V} L_v$ and $LBV = \cup_{v \in BV} L_v$. For any $u \in NV$ we put $K_u = LV \cup LBV$. It is easy to see that for any $\mathbf{x}'$ such that $x_i = x'_i$ for every $i \in LV \cup LBV$, the vertex u is k-vertex in the graph corresponding to $\mathbf{x}'$.

For $i \in LV \cup LBV$ we estimate $c_i(\mathbf{x}, \mathcal{K})$ by $2k+1$. Note that $|LV| \leqslant z + \frac{z}{k}$. We add additional $\frac{z}{k}$ variables because the vertices from V can have loops. We have $\deg w \leqslant k+1$ for $w \in BV \backslash V$ and $|BV \backslash V| \leqslant z/(k+1)$, therefore $|LBV \backslash LV| \leqslant z$. So we have

$$\sum_{i \in LV \cup LBV} c_i(\mathbf{x}, \mathcal{K}) \leqslant (2k+1)(|LBV \backslash LV| + |LV|) \leqslant (2k+1)\left(2z + \frac{z}{k}\right) \leqslant (4k+5)z.$$

Next we consider the set W of the remaining k-vertices. We have $|W| = q - z$. By the definition, for any $w \in W$ all the neighbors N_w of w have degree less than or equal to $k+1$.

For each $w \in W$ we consider $V_w = \{v_1, \ldots, v_w\}$, $V_w \subset N_w$, such that the number of edges adjacent to at least one of $v_i \in N_w$ and not adjacent to w is between k and $2k$. We can find such V_w since w is a k-vertex. We can choose $v_i \in N_w$ one by one, until the total number of 2-adjacent edges does not exceed k. But it cannot exceed $2k$ since $\deg v_i \leqslant k+1$ for $v_i \in N_w$. Denote by LV_w all the variables that lead to V_w.

Now for each $w \in W$ we put $K_w = LV_w \cup L_w$. Note that $|LV_w| \leqslant 2k$ and for $w \in W \backslash V$ we have $|L_w| \leqslant k+1$.

Now we can make the final estimate:

$$\sum_{i=1}^{n} c_i(\mathbf{x}, \mathcal{K}) = \sum_{i \in LV \cup LBV} c_i(\mathbf{x}, \mathcal{K}) + \sum_{i \in \{1, \ldots, n\} \backslash LV \cup LBV} c_i(\mathbf{x}, \mathcal{K}) \leqslant (4k+5)z +$$
$$+ \sum_{w \in W} |LV_w| + \sum_{w \in W \backslash V} |L_w| \leqslant (4k+5)z + 2k(q-z) + (k+1)(q-z) \leqslant (4k+5)q.$$

The fact that $\mathcal{K}$ is a stable set system follows from the construction. $\square$

3.4 Application of Talagrand's inequality

First we briefly review Talagrand's inequality (see e.g., [1]).

Let $\Omega = \prod_{i=1}^{n} \Omega_i$ be a product probability space with product measure. Suppose $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\sum_{i=1}^{n} \alpha_i^2 = 1$. We define the following distance between a set $A \subset \Omega$ and a point $\mathbf{x} \in \Omega$:

$$\operatorname{dist}(A, \mathbf{x}) = \max_{\alpha} \min_{\mathbf{y} \in A} \sum_{i \in I_{\mathbf{xy}}} \alpha_i,$$

where $I_{\mathbf{xy}} = \{i : x_i \neq y_i\}$.

For $t > 0$ we denote by A_t the set $\{\mathbf{x} : \operatorname{dist}(A, \mathbf{x}) \leqslant t\}$.

Theorem 8. (Talagrand's inequality) *For any $t > 0$ and $A \subset \Omega$ we have* $\mathsf{P}(A)(1-\mathsf{P}(A_t)) \leqslant e^{-\frac{t^2}{4}}$.

We use the inequality to derive the following theorem.

Theorem 9. *For $t > 0$, $k, s \in \mathbb{N}$, and $f(s)$ that satisfies the condition $f^2(s) > (2k+1)(4k+5)s$ we have* $\mathsf{P}\big(Y_n(k) \leqslant s - tf(s)\big)\,\mathsf{P}\big(Y_n(k) \geqslant s\big) \leqslant e^{-\frac{t^2}{4}}$.

Proof. The inequality is trivial for $tf(s) > s$, so we can assume w.l.o.g. that $tf(s) \leqslant s$. Since ξ_i are independent, we can apply Talagrand's inequality to the points $\mathbf{x}$ from the probability space Ω. Actually, all we need to prove is that for any $\mathbf{x}$ such that $Y_n(k,\mathbf{x}) \geqslant s$ we have $\mathbf{x} \notin A_t$, where $A = \{\mathbf{y} : Y_n(k,\mathbf{y}) \leqslant s - tf(s)\}$.

Suppose $Y_n(k,\mathbf{x}) = q \geqslant s$. Given $\mathbf{x}$ we fix a set system $\mathcal{K}$ as in Lemma 5. Then by Lemma 4 for any $\mathbf{y} \in A$ we have $q - s + tf(s) \leqslant Y_n(k,\mathbf{x}) - Y_n(k,\mathbf{y}) \leqslant \sum_{j\in I_{\mathbf{xy}}} c_j(\mathbf{x},\mathcal{K})$.

We define a suitable vector $\alpha = \alpha(\mathbf{x})$. Namely, $\alpha_i = \frac{c_i(\mathbf{x},\mathcal{K})}{\sqrt{\sum_{j=1}^n c_j^2(\mathbf{x},\mathcal{K})}}$. It is easy to see that $\sum \alpha_j^2 = 1$.

We have
$$\sum_{j=1}^n c_j^2(\mathbf{x},\mathcal{K}) \leqslant \max_j c_j(\mathbf{x},\mathcal{K}) \sum_{j=1}^n c_j(\mathbf{x},\mathcal{K}) \leqslant (2k+1)(4k+5)q.$$
In the last inequality we used Lemma 5 and the definition of $c_j(\mathbf{x},\mathcal{K})$.

Now we show that $\sum_{i\in I_{\mathbf{xy}}} \alpha_i > t$ for any $\mathbf{y} \in A$.
$$\sum_{i\in I_{\mathbf{xy}}} \alpha_i = \frac{\sum_{i\in I_{\mathbf{xy}}} c_i(\mathbf{x},\mathcal{K})}{\sqrt{\sum_{j=1}^n c_j^2(\mathbf{x},\mathcal{K})}} \geqslant \frac{q - s + tf(s)}{\sqrt{(2k+1)(4k+5)q}} \geqslant \frac{tf(s)}{\sqrt{(2k+1)(4k+5)s}} > t. \tag{1}$$
The second inequality holds since for $q \geqslant s, tf(s) \leqslant s$ we have $\frac{q-s+tf(s)}{q} \geqslant \frac{tf(s)}{s}$. The last inequality follows from the statement of the theorem.

From (1) we obtain that $\operatorname{dist}(A,\mathbf{x}) > t$, in other words, $\mathbf{x} \notin A_t$. □

We apply Theorem 9 with $t = 2\ln n$, $s = m(Y_n(k)) + t(\mathsf{E}Y_n(k))^{1-\varepsilon}$, $f(s) = (\mathsf{E}Y_n(k))^{1-\varepsilon}$. Here $m(Y_n(k))$ is the median of $Y_n(k)$, and, consequently, $\mathsf{P}\big(Y_n(k) \leqslant s - tf(s)\big) \geqslant 1/2$. Since for any random variable Z we have $m(Z) \leqslant 2\mathsf{E}Z$, it is easy to see that the conditions of Theorem 9 hold if
$$(\mathsf{E}Y_n(k))^{1-2\varepsilon} \geqslant 12(2k+1)^2 \ln n.$$
If ε is small enough then this inequality is a consequence of Corollary 1 and the conditions of Theorem 5.

We obtain that
$$\mathsf{P}\Big(Y_n(k) \geqslant m(Y_n(k)) + 2\ln n(\mathsf{E}Y_n(k))^{1-\varepsilon}\Big) \leqslant 2e^{-\frac{t^2}{4}} = \bar{o}(1/n),$$
and since $Y_n(k) \leqslant n$ for all k, we have
$$\mathsf{E}Y_n(k) \leqslant m(Y_n(k)) + 2\ln n(\mathsf{E}Y_n(k))^{1-\varepsilon} + \bar{o}(1).$$

Similarly we can derive that

$$\mathsf{P}\Big(Y_n(k) \leqslant m(Y_n(k)) - 2\ln n(\mathsf{E}Y_n(k))^{1-\varepsilon}\Big) \leqslant 2e^{-\frac{t^2}{4}} = \bar{o}(1/n)$$

and

$$\mathsf{E}Y_n(k) \geqslant m(Y_n(k)) - 2\ln n(\mathsf{E}Y_n(k))^{1-\varepsilon} - \bar{o}(1).$$

Consequently, for some $\delta > 0$ and all sufficiently large n we have $|\mathsf{E}Y_n(k) - m(Y_n(k))| \leqslant (\mathsf{E}Y_n(k))^{1-\delta}$. Therefore, for some $\varepsilon' > 0$

$$\mathsf{P}\left(|Y_n(k) - \mathsf{E}(Y_n(k))| > \mathsf{E}(Y_n(k))^{1-\varepsilon'}\right) = \bar{o}(1).$$

This concludes the proof of Theorem 5.

3.5 Proof of Theorem 6

We use the obvious fact that $X_n(k) = Y_n(k) - Y_n(k+1)$. Fix some $\epsilon' > 0$. First we want to apply Theorem 9 to $Y_n(k)$ and $Y_n(k+1)$. We argue as after the proof of Theorem 9. We put $f(s) = \frac{n^{1-\epsilon'}}{k^{1+a}}$, $t = 2\ln n$ and $s_1 = m(Y_n(k)) + tf(s)$, $s_2 = m(Y_n(k))$, $s_3 = m(Y_n(k+1)) + tf(s)$, $s_4 = m(Y_n(k+1))$.

We apply Theorem 9 to $Y_n(k)$ with s_1 and s_2, and to $Y_n(k+1)$ with s_3 and s_4 and obtain

$$\mathsf{P}\left(|Y_n(k) - \mathsf{E}(Y_n(k))| > \frac{n^{1-\epsilon'+\bar{o}(1)}}{k^{1+a}}\right) = \bar{o}(1),$$

$$\mathsf{P}\left(|Y_n(k+1) - \mathsf{E}(Y_n(k+1))| > \frac{n^{1-\epsilon'+\bar{o}(1)}}{k^{1+a}}\right) = \bar{o}(1),$$

provided

$$\frac{n^{2-2\epsilon'}}{k^{2+2a}} \geqslant \Theta\left(nk^{2-a}\right)\ln n.$$

It is easy to see that this holds if the conditions of Theorem 6 are satisfied for some $\delta > 0$. We have $|X_n(k) - \mathsf{E}(X_n(k))| \leqslant |Y_n(k) - \mathsf{E}(Y_n(k))| + |Y_n(k+1) - \mathsf{E}(Y_n(k+1))|$, so

$$\mathsf{P}\left(|X_n(k) - \mathsf{E}(X_n(k))| > \frac{n^{1-\epsilon'+\bar{o}(1)}}{k^{1+a}}\right) = \bar{o}(1).$$

Since $\frac{n^{1-\epsilon'+\bar{o}(1)}}{k^{1+a}} = \mathsf{E}(X_n(k))^{1-\epsilon}$ for some $\epsilon > 0$, this inequality completes the proof of Theorem 6.

3.6 Generalization to the case of arbitrary m

The proof of Theorem 9 can be modified to the case of the graph $H^n_{a,m}$. We present only the sketch of the argument. Suppose $m > 1$ is fixed. The number of variables changes from n to mn. The interpretation in terms of independent variables works for this case. Lemmas 3, 4, 5 hold for $m > 1$, but with some minor changes.

When we take a bundle of edges from a vertex v and move it to some vertex v', we can spoil not only the neighborhood of v, but also the vertex v'. Namely, suppose we change edges $(v, w_1), \dots, (v, w_l)$ to $(v', w_1), \dots, (v', w_l)$. If v' was a k-vertex and edges $(v, w_1), \dots, (v, w_l)$ were counted in the second degree of v', then the second degree of v' may decrease (this is impossible in the graph $H^n_{a,1}$ since $H^n_{a,1}$ is a tree). It is not difficult to see that we cannot spoil the other vertices.

Hence, we can formulate some analogs of Lemmas 3, 4.

Lemma 6. *For any* $\mathbf{x} = (x_1, \dots, x_n)$ *and* $i \in \{1, \dots, n\}$ *we have* $c(i, \mathbf{x}) \leqslant 2k + 2$.

We also have

$$c(i, \mathbf{x}) - 1 \leqslant \min\{2k + 1, C(i, \mathbf{x}, \mathcal{K})\} =: c_i(\mathbf{x}, \mathcal{K}).$$

Lemma 7. *Let* $\mathcal{K}$ *be a stable set collection. We have* $Y_n(k, \mathbf{x}) - Y_n(k, \mathbf{x}') \leqslant \sum_{j \in J}(c_j(\mathbf{x}, \mathcal{K}) + 1)$ *for any vector* $\mathbf{x}'$ *such that* $x'_i = x_i$ *for all* $i \in \{1, \dots, n\} \backslash J$.

Lemma 5 holds for $c_j(\mathbf{x}, \mathcal{K})$ and $m > 1$ without any changes.

The only thing left is to modify the proof of Theorem 9. We put $\alpha_i = \frac{c_i(\mathbf{x}, \mathcal{K}) + 1}{\sqrt{\sum_{j=1}^{mn}(c_j(\mathbf{x}, \mathcal{K}) + 1)^2}}$. Then

$$\sum_{j=1}^{mn}(c_j(\mathbf{x}, \mathcal{K}) + 1)^2 \leqslant \left(\max_j c_j(\mathbf{x}, \mathcal{K}) + 2\right) \sum_{j=1}^{mn} c_j(\mathbf{x}, \mathcal{K}) + mn \leqslant (2k + 3)(4k + 5)q + mn.$$

Finally,

$$\sum_{i \in I_{\mathbf{xy}}} \alpha_i = \frac{\sum_{i \in I_{\mathbf{xy}}}(c_i(\mathbf{x}, \mathcal{K}) + 1)}{\sqrt{\sum_{j=1}^{mn}(c_j(\mathbf{x}, \mathcal{K}) + 1)^2}} \geqslant \frac{q - s + tf(s)}{\sqrt{(2k + 3)(4k + 5)q + mn}} \geqslant \frac{tf(s)}{\sqrt{(2k + 3)(4k + 5)s + mn}} > t,$$

if $f^2(s) > (2k + 3)(4k + 5)s + mn$. So we can formulate an analog of Theorem 9.

Theorem 10. *For* $t > 0$, $m, k, s \in \mathbb{N}$, *and* $f(s)$ *that satisfies the condition* $f^2(s) > (2k + 3)(4k + 5)s + mn$ *we have*

$$\mathsf{P}\big(Y^m_n(k) \leqslant s - tf(s)\big)\, \mathsf{P}\big(Y^m_n(k) \geqslant s\big) \leqslant e^{-\frac{t^2}{4}}.$$

Finally, arguing as after the proof of Theorem 9, one can see that Theorem 7 follows from Theorem 10 and Conjecture 1.

4 Estimation of $\mathsf{E}Y_n(k)$

We need the following notation. Let X be a function on n (the number of vertices), l (the first degree we are interested in), k (the second degree we are interested in); then denote by $\theta(X)$ some function on n, l, k such that $|\theta(X)| < X$.

4.1 Proof of Theorem 4

It follows from the definition of $H_{a,1}^n$ that $N_n(l,0) = N_n(0,k) = 0$. Indeed, since we have no vertices of degree 0, we see that $N_n(0,k) = 0$. Since vertices with loops are not counted in $N_n(l,k)$, we have no vertices of second degree 0 and $N_n(l,0) = 0$. Therefore, we have $\mathsf{E}N_n(l,0) = \mathsf{E}N_n(0,k) = 0$. We want to prove that there exists such constant C that

$$\mathsf{E}N_n(l,k) = c(l,k)\,(n + \theta(n,l,k)),$$

where $|\theta(n,l,k)| < C(l+k)^{1+a}$.

Let us demonstrate that $\mathsf{E}N_n(1,k) = c(1,k)\ (n + \theta\,(Ck^{1+a}))$. We shall use induction on k. For $k = 0$ there is nothing to prove.

Assume that for $j < k$ we have

$$\mathsf{E}N_n(1,j) = c(1,j)\,(n + \theta\left(Cj^{1+a}\right)).$$

Denote by $N_i(l)$ the number of vertices with degree l in $H_{a,1}^i$. We use induction on i and the equality

$$\begin{aligned}\mathsf{E}(N_{i+1}(1,k)|N_i(1,k), N_i(1,k-1), N_i(k)) = \\ = N_i(1,k)\left(1 - \frac{k+2a}{(a+1)i+a}\right) + \frac{(k-1+a)N_i(1,k-1)}{(a+1)i+a} + \frac{(k-1+a)\,N_i(k)}{(a+1)i+a}.\end{aligned} \quad (2)$$

Let us explain this equality. Suppose we constructed $H_{a,1}^i$. We add one vertex and one edge. There are $N_i(1,k)$ vertices with degree 1 and with second degree k in $H_{a,1}^i$. The probability that we "spoil" one of these vertices is $\frac{k+2a}{(a+1)i+a}$. We also have $N_i(1,k-1)$ vertices with degree 1 and with second degree $k-1$. The probability that one of these vertices has degree 1 and second degree k in $H_{a,1}^{i+1}$ is $\frac{k-1+a}{(a+1)i+a}$. Finally, with probability equal to $\frac{k-1+a}{(a+1)i+a}$ the vertex $i+1$ has necessary degrees in $H_{a,1}^{i+1}$. From (2) we obtain

$$\mathsf{E}N_{i+1}(1,k) = \mathsf{E}N_i(1,k)\left(1 - \frac{k+2a}{(a+1)i+a}\right) + \frac{(k-1+a)\mathsf{E}N_i(1,k-1)}{(a+1)i+a} + \frac{(k-1+a)\,\mathsf{E}N_i(k)}{(a+1)i+a}. \quad (3)$$

Note that if we have at least one vertex with first degree 1 and second degree k in $H_{a,1}^i$, then we have at least k edges in this graph. Therefore $\mathsf{E}N_i(1,k) = 0$ when $i < k$. Consider the case $i = k$. First, note that

$$\mathsf{E}N_k(1,k) \geqslant c(1,k)\left(k + \theta\left(Ck^{1+a}\right)\right)$$

with some C. For a finite number of small k we can find a constant C such that

$$\mathsf{E}N_k(1,k) = c(1,k)\left(k + \theta\left(Ck^{1+a}\right)\right).$$

Using (3), Lemma 1, and the assumptions of the theorem we get

$$\mathsf{E}N_k(1,k) = \mathsf{E}N_{k-1}(1,k-1)\frac{k-1+a}{ak+k-1} + M^1_{k-1}(k)\frac{k-1+a}{ak+k-1} =$$

$$= c(1,k-1)\frac{k-1+a}{ak+k-1}\left(k-1+\theta\left(C\left(k-1\right)^{1+a}\right)\right) + c(k)\frac{k-1+a}{ak+k-1}\left(k-1+\theta\left(C_1k^{1+a}\right)\right) =$$

$$= c(1,k)\frac{(k+3a+1)(k-1)}{ak+k-1} + c(1,k-1)\frac{k-1+a}{ak+k-1}\theta\left(C\left(k-1\right)^{1+a}\right) + c(k)\frac{k-1+a}{ak+k-1}\theta\left(C_1k^{1+a}\right) =$$

$$= kc(1,k) + \frac{3ak+k-3a-1-ak^2}{ak+k-1}c(1,k) + c(1,k-1)\frac{k-1+a}{ak+k-1}\theta\left(C\left(k-1\right)^{1+a}\right) +$$

$$+c(k)\frac{k-1+a}{ak+k-1}\theta\left(C_1k^{1+a}\right) \leqslant kc(1,k) + \frac{(3ak+k-3a-1-ak^2)(a+k-1)}{(ak+k-1)(k+3a+1)}c(1,k-1)+$$

$$+\frac{(3ak+k-3a-1-ak^2)(a+k-1)}{(ak+k-1)(k+3a+1)}c(k) + c(1,k-1)\frac{C(k-1+a)}{ak+k-1}\left(k-1\right)^{1+a} +$$

$$c(k)\frac{k-1+a}{ak+k-1}C_1k^{1+a} \leqslant kc(1,k) + \frac{C(a+k-1)}{k+3a+1}c(1,k-1)k^{1+a} + \frac{C(a+k-1)}{k+3a+1}c(k)k^{1+a}.$$

This holds for big values of k. Indeed,

$$\frac{(3ak+k-3a-1-ak^2)}{(ak+k-1)(k+3a+1)} + \frac{C\left(k-1\right)^{1+a}}{ak+k-1} \leqslant \frac{C}{k+3a+1}k^{1+a},$$

if k and C are big enough.

Consider the case $i \geqslant k$. Using (3), Lemma 1, and the inductive assumption we get

$$\mathsf{E}N_{i+1}(1,k) = \mathsf{E}N_i(1,k)\left(1-\frac{k+2a}{(a+1)i+a}\right) + \mathsf{E}N_i(1,k-1)\frac{k-1+a}{(a+1)i+a} + M^1_i(k)\frac{k-1+a}{(a+1)i+a} =$$

$$= c(1,k)\left(i+\theta\left(Ck^{1+a}\right)\right)\left(1-\frac{k+2a}{(a+1)i+a}\right) + c(1,k-1)\left(i+\theta\left(C(k-1)^{1+a}\right)\right)\frac{k-1+a}{(a+1)i+a}+$$

$$+c(k)\left(i+\theta_1\left(C_1k^{1+a}\right)\right)\frac{k-1+a}{(a+1)i+a} = c(1,k)(i+1)-$$

$$-c(1,k)\frac{i(k+3a+1)+a}{(a+1)i+a} + c(1,k)\theta\left(Ck^{1+a}\right)\left(1-\frac{k+2a}{(a+1)i+a}\right) + c(1,k-1)i\frac{k-1+a}{(a+1)i+a}+$$

$$+c(1,k-1)\theta\left(C(k-1)^{1+a}\right)\frac{k-1+a}{(a+1)i+a} + c(k)i\frac{k-1+a}{(a+1)i+a} + c(k)\theta_1\left(C_1k^{a+1}\right)\frac{k-1+a}{(a+1)i+a} =$$

$$= c(1,k)(i+1) + c(1,k)\theta\left(Ck^{1+a}\right)\left(1-\frac{k+2a}{(a+1)i+a}\right) - \frac{(k-1+a)ac(1,k-1)}{((a+1)i+a)(k+3a+1)}-$$

$$-\frac{(k-1+a)ac(k)}{((a+1)i+a)(k+3a+1)} + c(1,k-1)\theta\left(C(k-1)^{1+a}\right)\frac{k-1+a}{(a+1)i+a} + c(k)\theta_1\left(C_1k^{a+1}\right)\frac{k-1+a}{(a+1)i+a}.$$

We want to prove that there exists a constant C such that

$$c(1,k)Ck^{1+a}\frac{k+2a}{(a+1)i+a} \geqslant \frac{(k-1+a)ac(1,k-1)}{((a+1)i+a)(k+3a+1)} + \frac{(k-1+a)ac(k)}{((a+1)i+a)(k+3a+1)}+$$

$$+c(1,k-1)C(k-1)^{1+a}\frac{k-1+a}{(a+1)i+a}+c(k)C_1k^{1+a}\frac{k-1+a}{(a+1)i+a}.$$

It is sufficient to prove that the following inequalities hold:

$$c(1,k-1)Ck^{1+a}\frac{(k+2a)(k-1+a)}{((a+1)i+a)(k+3a+1)}\geqslant\frac{(k-1+a)ac(1,k-1)}{((a+1)i+a)(k+3a+1)}+c(1,k-1)C(k-1)^{1+a}\frac{k-1+a}{(a+1)i+a}$$

and

$$c(k)Ck^{1+a}\frac{(k+2a)(k-1+a)}{((a+1)i+a)(k+3a+1)}\geqslant\frac{(k-1+a)ac(k)}{((a+1)i+a)(k+3a+1)}+c(k)C_1k^{1+a}\frac{k-1+a}{(a+1)i+a}.$$

Or

$$Ck^{1+a}(k+2a)\geqslant a+C(k-1)^{1+a}(k+3a+1),$$
$$Ck^{1+a}(k+2a)\geqslant a+C_1k^{1+a}(k+3a+1).$$

Note that

$$k^{1+a}(k+2a)-(k-1)^{1+a}(k+3a+1)=$$
$$=k^{1+a}(k+2a)-(k^{1+a}-(1+a)k^a+\frac{a(a+1)}{2}k^{a-1}+O(k^{a-2}))(k+3a+1)=\frac{(5a+2)(a+1)}{2}k^a+O(k^{a-1}).$$

For big values of k there exists a constant C such that

$$C(k^{1+a}(k+2a)-(k-1)^{1+a}(k+3a+1))\geqslant a.$$

But we can not choose a constant C if $k^{1+a}(k+2a)\leqslant(k-1)^{1+a}(k+3a+1)$. There is a finite number of k with $\frac{(5a+2)(a+1)}{2}k^a+O(k^{a-1})\leqslant 0$. For such k we want to prove that $\mathsf{E}N_n(1,k)=c(1,k)\ (n+O\left(f(k)\right))$ with some function $f(k)$. Using the method above we obtain the same inequalities:

$$f(k)(k+2a)\geqslant a+f(k-1)(k+3a+1),$$
$$f(k)(k+2a)\geqslant a+C_1k^{1+a}(k+3a+1).$$

There exists a function f such that the inequalities hold. This completes the proof for $\mathsf{E}N_n(1,k)$.

Consider the case $l>1$. Assume that we already proved that

$$\mathsf{E}N_n(i,j)=c(i,j)\ \left(n+\theta\left(C(i+j)^{1+a}\right)\right)$$

for all i and j, such that $i<l$, $j\leqslant k$ or $i\leqslant l$, $j<k$.

We use the following equality, which is similar to (3):

$$\mathsf{E}N_{i+1}(l,k)=\mathsf{E}N_i(l,k)\left(1-\frac{l(1+a)+k+a-1}{(a+1)i+a}\right)+$$
$$+\frac{(l-2+a)\,\mathsf{E}N_i(l-1,k)}{(a+1)i+a}+\frac{(k+al-1)\mathsf{E}N_i(l,k-1)}{(a+1)i+a}. \quad (4)$$

Note that if we have at least one vertex with first degree l and second degree k in $H^i_{a,1}$ (without a loop), then we have at least $l+k-1$ edges in this graph. Therefore $\mathsf{E}N_i(l,k)=0$ when $i<l+k-1$. Consider the case $i=l+k-1$. It is sufficient to prove that

$$\mathsf{E}N_{l+k-1}(l,k)\leqslant Cc(l,k)(l+k)$$

with some C. For any finite number of small l and k we can easily find a constant C such that

$$\mathsf{E}N_{l+k-1}(l,k) \leqslant Cc(l,k)(l+k).$$

Using (4), we get

$$\mathsf{E}N_{l+k-1}(l,k) = \frac{(l-2+a)\,\mathsf{E}N_{l+k-2}(l-1,k)}{(a+1)(l+k-2)+a} + \frac{(k+al-1)\mathsf{E}N_{l+k-2}(l,k-1)}{(a+1)(l+k-2)+a} \leqslant$$

$$\leqslant Cc(l-1,k)\frac{(l-2+a)\,(l+k-1)}{(a+1)(l+k-2)+a} + Cc(l,k-1)\frac{(k+al-1)\,(l+k-1)}{(a+1)(l+k-2)+a} \leqslant$$

$$\leqslant Cc(l-1,k)\frac{(l-2+a)(l+k)}{l+al+k+2a} + Cc(l,k-1)\frac{(al+k-1)(l+k)}{l+al+k+2a}.$$

The last inequality holds if k is big enough.

We also need to consider a finite number of small k. First we show that for any finite number of small k we have

$$c(l,k) = \Omega\left(\frac{l^{k-4+\frac{a^2}{a+1}}}{(1+a)^l}\right).$$

Indeed, from the recurrent relation we obtain

$$c(l,1) = c(l-1,1)\frac{l-2+a}{(a+1)(l+1+a/(a+1))}.$$

Therefore

$$c(l,1) = \Omega\left(\frac{\Gamma(l-1+a)}{(a+1)^l\Gamma(l+2+a/(a+1))}\right) = \Omega\left(\frac{l^{-3+\frac{a^2}{a+1}}}{(1+a)^l}\right).$$

Here we used Statement 1 from Subsection 4.2. For $k \geqslant 2$ we have

$$c(l,k) = c(l,k-1)\frac{al+k-1}{l(1+a)+k+2a} + c(l-1,k)\frac{l-2+a}{l(1+a)+k+2a}.$$

It is sufficient to prove that there exists a positive function $f(k)$ such that

$$f(k)(l(1+a)+k+2a)l^{k-4+\frac{a^2}{a+1}} \leqslant f(k-1)(al+k-1)l^{k-5+\frac{a^2}{a+1}} + f(k)(l-2+a)(a+1)(l-1)^{k-4+\frac{a^2}{a+1}},$$

$$f(k)(l(1+a)+k+2a)\left(l^{k-4+\frac{a^2}{a+1}} - (l-1)^{k-4+\frac{a^2}{a+1}}\right) + f(k)(3a+k-a^2+2)(l-1)^{k-4+\frac{a^2}{a+1}} \leqslant$$

$$\leqslant f(k-1)(al+k-1)l^{k-5+\frac{a^2}{a+1}}.$$

The last inequality holds for some positive function $f(k)$.

So we want to prove that

$$\mathsf{E}N_{l+k-1}(l,k) = O\left(\frac{l^{k-3+\frac{a^2}{a+1}}}{(1+a)^l}\right).$$

Suppose that we have a graph on $l+k-1$ vertices and a vertex t has first degree l and second degree k. Then one edge from this vertex goes to the vertex 1, $l-1$ vertices send edges to t, and $k-2$ vertices send edges to the neighbors of t. There are $\binom{l+k-2}{k-2}$ ways to choose our vertex t and its neighbors. In each case the probability of these neighbors to send edges to the vertex t equals

$$O\left(\frac{(a(a+1)\dots(a+l-2))}{(3a+2)(4a+3)\dots(l(a+1)+a)}\right) = O\left(\frac{\Gamma(a+l-1)}{(a+1)^l\Gamma(l+1+a/(a+1))}\right) = O\left(\frac{l^{-2+\frac{a^2}{a+1}}}{(a+1)^l}\right),$$

so

$$\mathsf{E}N_{l+k-1}(l,k) = O\left(\frac{l^{k-4+\frac{a^2}{a+1}}}{(a+1)^l}\right).$$

This conclues the case $i = l+k-1$.

For $i \geqslant l+k-1$ we have

$$\mathsf{E}N_{i+1}(l,k) = \mathsf{E}N_i(l,k)\left(1 - \frac{l(1+a)+k+a-1}{(a+1)i+a}\right) + \frac{(l-2+a)\,\mathsf{E}N_i(l-1,k)}{(a+1)i+a} +$$

$$+\frac{(k+al-1)\mathsf{E}N_i(l,k-1)}{(a+1)i+a} = c(l,k)\left(i+\theta\left(C(l+k)^{1+a}\right)\right)\left(1-\frac{l(1+a)+k+a-1}{(a+1)i+a}\right)+$$

$$+c(l-1,k)\left(i+\theta\left(C(l+k-1)^{1+a}\right)\right)\frac{(l-2+a)}{(a+1)i+a}+c(l,k-1)\left(i+\theta\left(C(l+k-1)^{1+a}\right)\right)\frac{(k+al-1)}{(a+1)i+a} =$$

$$= c(l,k)i - c(l,k)i\frac{l(1+a)+k+a-1}{(a+1)i+a} + c(l,k)\theta\left(C(l+k)^{1+a}\right)\left(1-\frac{l(1+a)+k+a-1}{(a+1)i+a}\right)+$$

$$+c(l-1,k)i\frac{(l-2+a)}{(a+1)i+a} + c(l,k-1)i\frac{(k+al-1)}{(a+1)i+a}+$$

$$+c(l-1,k)\theta\left(C(l+k-1)^{1+a}\right)\frac{(l-2+a)}{(a+1)i+a} + c(l,k-1)\theta\left(C(l+k-1)^{1+a}\right)\frac{(k+al-1)}{(a+1)i+a} =$$

$$= c(l,k)(i+1) - c(l,k)\frac{il(1+a)+ik+2ia+a}{(a+1)i+a} + c(l-1,k)i\frac{(l-2+a)}{(a+1)i+a}+$$

$$+c(l,k-1)i\frac{(k+al-1)}{(a+1)i+a} + c(l,k)\theta\left(C(l+k)^{1+a}\right)\left(1-\frac{l(1+a)+k+a-1}{(a+1)i+a}\right)+$$

$$+c(l-1,k)\theta\left(C(l+k-1)^{1+a}\right)\frac{(l-2+a)}{(a+1)i+a} + c(l,k-1)\theta\left(C(l+k-1)^{1+a}\right)\frac{(k+al-1)}{(a+1)i+a} =$$

$$= c(l,k)(i+1) - \frac{a(k+al-1)c(l,k-1)}{((a+1)i+a)(l(1+a)+k+2a)} - \frac{a(l-2+a)c(l-1,k)}{((a+1)i+a)(l(1+a)+k+2a)}+$$

$$+c(l,k)\theta\left(C(l+k)^{1+a}\right)\left(1-\frac{l(1+a)+k+a-1}{(a+1)i+a}\right)+$$

$$+c(l-1,k)\theta\left(C(l+k-1)^{1+a}\right)\frac{(l-2+a)}{(a+1)i+a}+c(l,k-1)\theta\left(C(l+k-1)^{1+a}\right)\frac{(k+al-1)}{(a+1)i+a}.$$

We want to prove the following inequality:

$$Cc(l,k)\left((l+k)^{1+a}\right)\frac{l(1+a)+k+a-1}{(a+1)i+a}\geqslant$$

$$\geqslant\frac{a(k+al-1)c(l,k-1)}{((a+1)i+a)(l(1+a)+k+2a)}+\frac{a(l-2+a)c(l-1,k)}{((a+1)i+a)(l(1+a)+k+2a)}+$$

$$+Cc(l-1,k)\left((l+k-1)^{1+a}\right)\frac{(l-2+a)}{(a+1)i+a}+Cc(l,k-1)\left((l+k-1)^{1+a}\right)\frac{(k+al-1)}{(a+1)i+a}.$$

It is sufficient to show that the following inequalities hold

$$Cc(l,k-1)(l+k)^{1+a}\frac{(l(1+a)+k+a-1)(k+al-1)}{((a+1)i+a)(l(1+a)+k+2a)}\geqslant$$

$$\geqslant\frac{a(k+al-1)c(l,k-1)}{((a+1)i+a)(l(1+a)+k+2a)}+Cc(l,k-1)(l+k-1)^{1+a}\frac{(k+al-1)}{(a+1)i+a}$$

and

$$Cc(l-1,k)(l+k)^{1+a}\frac{(l(1+a)+k+a-1)(l-2+a)}{((a+1)i+a)(l(1+a)+k+2a)}\geqslant$$

$$\geqslant\frac{a(l-2+a)c(l-1,k)}{((a+1)i+a)(l(1+a)+k+2a)}+Cc(l-1,k)(l+k-1)^{1+a}\frac{(l-2+a)}{(a+1)i+a}.$$

In other words

$$C(l+k)^{1+a}(l(1+a)+k+a-1)(k+al-1)\geqslant$$

$$\geqslant a(k+al-1)+C(l+k-1)^{1+a}(k+al-1)(l(1+a)+k+2a)$$

and

$$C(l+k)^{1+a}(l(1+a)+k+a-1)(l-2+a)\geqslant$$

$$\geqslant a(l-2+a)+C(l+k-1)^{1+a}(l-2+a)(l(1+a)+k+2a).$$

To prove both inequalities we make the following transformations:

$$(l+k)^{1+a}(l(1+a)+k+a-1)-(l+k-1)^{1+a}(l(1+a)+k+2a)=$$

$$=(l+k)^{1+a}(l(1+a)+k+a-1)-((l+k)^{1+a}-(1+a)(l+k)^{a}+\frac{a(1+a)}{2}(l+k)^{a-1}+$$

$$+O\left((l+k)^{a-2}\right))\left(l(1+a)+k+2a\right)=-(l+k)^{1+a}(1+a)+(1+a)(l+k)^{a}(l(1+a)+k+2a)-$$

$$-\frac{a(1+a)}{2}(l+k)^{a-1}(l(1+a)+k+2a)+O\left((l+k)^{a-2}\right)(l(1+a)+k+2a)=$$

$$=(l+k)^{a-1}(1+a)\left(al^2+alk+2al+2ak-\frac{a(1+a)}{2}l-\frac{a}{2}k-\frac{2a^2}{2}\right)+O\left((l+k)^{a-2}\right)(l(1+a)+k+2a)=$$

$$=(l+k)^{a-1}(1+a)\left(al^2+alk+\frac{3}{2}al+\frac{3}{2}ak-\frac{1}{2}a^2l-a^2\right)+O\left((l+k)^{a-2}\right)(l(1+a)+k+2a).$$

If l or k is large enough, then there exists a constant C such that

$$C(l+k)^{a-1}(1+a)\left(al^2+alk+\frac{3}{2}al+\frac{3}{2}ak-\frac{1}{2}a^2l-a^2\right)+O\left((l+k)^{a-2}\right)(l(1+a)+k+2a)\geqslant a.$$

Finally, we need to consider the finite number of small l and k. We want to find some function $f(l,k)$ such that

$$f(l,k)c(l,k)\left((l+k)^{1+a}\right)\frac{l(1+a)+k+a-1}{(a+1)i+a}\geqslant$$

$$\geqslant\frac{a(k+al-1)c(l,k-1)}{((a+1)i+a)(l(1+a)+k+2a)}+\frac{a(l-2+a)c(l-1,k)}{((a+1)i+a)(l(1+a)+k+2a)}+$$
$$+f(l-1,k)c(l-1,k)\frac{(l-2+a)}{(a+1)i+a}+f(l,k-1)c(l,k-1)\frac{(k+al-1)}{(a+1)i+a}.$$

Such function $f(l,k)$ exists. This concludes the proof of Theorem 4.

4.2 Proof of Theorem 2

In this proof we shall use the following statement.

Statement 1. *For $t>0$ and fixed $a>0$*

$$\frac{\Gamma(t+a)}{\Gamma(t)}=t^a\left(1+O(1/t)\right).$$

Proof. From Stirling's formula we obtain

$$\frac{\Gamma(t+a)}{\Gamma(t)}=\sqrt{\frac{t}{t+a}}\frac{(t+a)^a}{e^a}\left(\frac{t+a}{t}\right)^t(1+1/t).$$

It is easy to check that

$$t\ln\left(1+\frac{a}{t}\right)=a+O(1/t).$$

So

$$\left(1+\frac{a}{t}\right)^t=e^a(1+O(1/t)).$$

We obtain

$$\frac{\Gamma(t+a)}{\Gamma(t)}=\sqrt{\frac{t}{t+a}}(t+a)^a\left(1+O(1/t)\right)=t^a\left(1+O(1/t)\right).$$

□

4.2.1 Estimation of $c(1,k)$

Lemma 8.

$$c(1,k) = \frac{\Gamma(2a+1)(1+O(1/k))}{\Gamma(a)k^{a+1}}.$$

Proof. As we know

$$c(k) = \frac{\mathrm{B}(k-1+a, a+2)}{\mathrm{B}(a, a+1)} = \frac{(a+1)\Gamma(2a+1)\Gamma(k-1+a)}{\Gamma(a)\Gamma(k+1+2a)}.$$

Using the recurrent relation

$$c(1,k) = c(1,k-1)\frac{a+k-1}{k+3a+1} + c(k)\frac{a+k-1}{k+3a+1}$$

we obtain

$$c(1,k) = \sum_{j=1}^{k} \frac{c(j)(a+j-1)\dots(a+k-1)}{(j+3a+1)\dots(k+3a+1)} =$$

$$= \frac{(a+1)\Gamma(2a+1)}{\Gamma(a)} \sum_{j=1}^{k} \frac{\Gamma(j-1+a)(a+j-1)\dots(a+k-1)}{\Gamma(j+1+2a)(j+3a+1)\dots(k+3a+1)} =$$

$$= \frac{(a+1)\Gamma(2a+1)\Gamma(a+k)}{\Gamma(a)\Gamma(k+3a+2)} \sum_{j=1}^{k} \frac{\Gamma(j+3a+1)}{\Gamma(j+1+2a)} =$$

$$= \frac{(a+1)\Gamma(2a+1)\Gamma(a+k)}{\Gamma(a)\Gamma(k+3a+2)} \sum_{j=1}^{k} j^a(1+O(1/j)) = \frac{\Gamma(2a+1)k^{a+1}(1+O(1/k))}{\Gamma(a)k^{2a+2}} =$$

$$= \frac{\Gamma(2a+1)(1+O(1/k))}{\Gamma(a)k^{a+1}}.$$

□

4.2.2 Sum of $c(l,k)$

We want to estimate the sum $\sum_{l=1}^{\infty} c(l,k)$. First let us prove that the series $\sum_{l=1}^{\infty} l^N c(l,k)$ converges for all N and k.

The inequality

$$c(l,k) \leqslant \tilde{C}\frac{p^k}{(1+q)^l}$$

holds for any $p > 1$ and $q = \min\{a, 1\}\frac{(p-1)}{p}$. Here we choose $\tilde{C}$ so that $\tilde{C}\frac{p^k}{1+ap} \geqslant c(1,k)$ for any k. We need to prove that

$$\frac{p^k}{(1+q)^l}(l+al+k+2a) \geqslant \frac{p^{k-1}}{(1+q)^l}(al+k-1) + \frac{p^k}{(1+q)^{l-1}}(l-2+a),$$

We make some transformations:

$$p(l+al+k+2a) \geqslant (al+k-1) + p(1+q)(l-2+a),$$

$$p(al+k+a+2) \geqslant (al+k-1) + pq(l-2+a),$$
$$al+k+a+2 \geqslant \min\{a,1\}(l-2+a).$$

The last inequality holds. Therefore $c(l,k) \leqslant \tilde{C}\frac{p^k}{(1+q)^l}$ and $\sum_{l=1}^{\infty} l^N c(l,k)$ converges.

For $l \geqslant 2$ and any $c \geqslant 0$ we have

$$c(l,k)(l(1+a)+k+2a)\frac{\Gamma(l+a+c)}{\Gamma(l+a-1)} = c(l,k-1)\frac{\Gamma(l+a+c)}{\Gamma(l+a-1)}(al+k-1)+c(l-1,k)(l-2+a)\frac{\Gamma(l+a+c)}{\Gamma(l+a-1)}.$$

Therefore,

$$\sum_{l=2}^{\infty} c(l,k)(l(1+a)+k+2a)\frac{\Gamma(l+a+c)}{\Gamma(l+a-1)} = \sum_{l=2}^{\infty} c(l,k-1)(al+k-1)\frac{\Gamma(l+a+c)}{\Gamma(l+a-1)}+\sum_{l=1}^{\infty} c(l,k)\frac{\Gamma(l+a+c+1)}{\Gamma(l+a-1)},$$

$$\sum_{l=2}^{\infty} c(l,k)(al+k+a-c)\frac{\Gamma(l+a+c)}{\Gamma(l+a-1)} = \sum_{l=2}^{\infty} c(l,k-1)(al+k-1)\frac{\Gamma(l+a+c)}{\Gamma(l+a-1)}+c(1,k)\frac{\Gamma(a+c+2)}{\Gamma(a)}.$$

Consider the function

$$f_c(k) = \frac{\Gamma(k+a-c)}{\Gamma(k)} = k^{a-c}(1+O(1/k)).$$

It is easy to see that

$$\frac{f_c(k+1)}{f_c(k)} = 1 + \frac{a-c}{k}.$$

We have

$$\sum_{j=1}^{k}\sum_{l=2}^{\infty} c(l,j)(al+j+a-c)\frac{\Gamma(l+a+c)}{\Gamma(l+a-1)}f_c(j) =$$

$$= \sum_{j=1}^{k}\sum_{l=2}^{\infty} c(l,j-1)(al+j-1)\frac{\Gamma(l+a+c)}{\Gamma(l+a-1)}f_c(j) + \sum_{j=1}^{k} c(1,j)\frac{\Gamma(a+c+2)}{\Gamma(a)}f_c(j),$$

$$\sum_{j=1}^{k}\sum_{l=2}^{\infty} c(l,j)(al+j+a-c)\frac{\Gamma(l+a+c)}{\Gamma(l+a-1)}f_c(j) =$$

$$= \sum_{j=1}^{k-1}\sum_{l=2}^{\infty} c(l,j)(al+j)\frac{\Gamma(l+a+c)}{\Gamma(l+a-1)}f_c(j)\left(1+\frac{a-c}{j}\right) + \sum_{j=1}^{k} c(1,j)\frac{\Gamma(a+c+2)}{\Gamma(a)}f_c(j),$$

$$\sum_{l=2}^{\infty} c(l,k)(al+k+a-c)\frac{\Gamma(l+a+c)}{\Gamma(l+a-1)}f_c(k) =$$

$$= \sum_{j=1}^{k-1}\sum_{l=2}^{\infty} c(l,j)\frac{\Gamma(l+a+c)}{\Gamma(l+a-1)}\frac{al(a-c)}{j}f_c(j) + \sum_{j=1}^{k} c(1,j)\frac{\Gamma(a+c+2)}{\Gamma(a)}f_c(j).$$

If $c \geqslant a$ then taking into account Lemma 8 and the above-mentioned asymptotics for $f_c(j)$ we have

$$\sum_{l=2}^{\infty} c(l,k)(al+k+a-c)\frac{\Gamma(l+a+c)}{\Gamma(l+a-1)}f_c(k) \leqslant \sum_{j=1}^{k} c(1,j)\frac{\Gamma(a+c+2)}{\Gamma(a)}f_c(j) = O(1).$$

Hence,

$$\sum_{l=2}^{\infty} c(l,k)\frac{\Gamma(l+a+c)}{\Gamma(l+a-1)}f_c(k) = O(1/k).$$

We want to prove that for any $0 \leqslant c < a+1$ the following equality holds:

$$\sum_{l=2}^{\infty} c(l,k)\frac{\Gamma(l+a+c)}{\Gamma(l+a-1)}f_c(k) = O\left(\frac{(\ln k)^{\lceil a-c\rceil}}{k}\right). \tag{5}$$

We have already proved this statement for $a \leqslant c < a+1$.

Suppose that for $c' \geqslant 1$ we have

$$\sum_{l=2}^{\infty} c(l,k)\frac{\Gamma(l+a+c')}{\Gamma(l+a-1)}f_{c'}(k) = O\left(\frac{(\ln k)^{\lceil a-c'\rceil}}{k}\right).$$

Then

$$\sum_{l=2}^{\infty} c(l,k)(al+k+a-c'+1)\frac{\Gamma(l+a+c'-1)}{\Gamma(l+a-1)}f_{c'-1}(k) =$$

$$= \sum_{j=1}^{k-1}\sum_{l=2}^{\infty} c(l,j)\frac{\Gamma(l+a+c'-1)}{\Gamma(l+a-1)}\frac{al(a-c'+1)}{j}(j+a-c')f_{c'}(j) + \sum_{j=1}^{k} c(1,j)\frac{\Gamma(a+c'+1)}{\Gamma(a)}f_{c'-1}(j) =$$

$$= O\left(\sum_{j=1}^{k-1}\frac{(\ln k)^{\lceil a-c'\rceil}}{j}\right) = O\left((\ln k)^{\lceil a-c'+1\rceil}\right).$$

We proved (5). In particular,

$$\sum_{l=2}^{\infty} c(l,k)\frac{\Gamma(l+a)}{\Gamma(l+a-1)}f_0(k) = \sum_{l=2}^{\infty} c(l,k)(l+a-1)f_0(k) = O\left(\frac{(\ln k)^{\lceil a\rceil}}{k}\right). \tag{6}$$

Put $x_k = \sum_{l=2}^{\infty} c(l,k)$. For $l \geqslant 2$

$$c(l,k)(l(1+a)+k+2a) = c(l,k-1)(al+k-1) + c(l-1,k)(l-2+a).$$

So

$$\sum_{l=2}^{\infty} c(l,k)(l(1+a)+k+2a) = \sum_{l=2}^{\infty} c(l,k-1)(al+k-1) + \sum_{l=1}^{\infty} c(l,k)(l-1+a),$$

$$\sum_{l=2}^{\infty} c(l,k)(al+k+a+1) = \sum_{l=2}^{\infty} c(l,k-1)(al+k-1) + ac(1,k),$$

$$(k+a+1)x_k = (k-1)x_{k-1} + ac(1,k) + a\sum_{l=2}^{\infty} l(c(l,k-1)-c(l,k)).$$

We have

$$(k+a+1)f_{-1}(k)x_k = (k-1)f_{-1}(k)x_{k-1} + af_{-1}(k)c(1,k) + af_{-1}(k)\sum_{l=2}^{\infty} l(c(l,k-1)-c(l,k)),$$

$$\sum_{j=1}^{k}(j+a+1)f_{-1}(j)x_j = \sum_{j=1}^{k-1} f_{-1}(j)(j+a+1)x_j + \sum_{j=1}^{k} af_{-1}(j)c(1,j) + \sum_{j=1}^{k} af_{-1}(j)\sum_{l=2}^{\infty} l(c(l,j-1)-c(l,j)),$$

$$(k+a+1)f_{-1}(k)x_k = a\sum_{j=1}^{k} f_{-1}(j)c(1,j) + a\sum_{j=1}^{k} f_{-1}(j)\sum_{l=2}^{\infty} l(c(l,j-1)-c(l,j)).$$

$$f_{-1}(k)(k+a+1)x_k = a\sum_{j=1}^{k} f_{-1}(j)c(1,j) + a(a+1)\sum_{j=1}^{k-1}\frac{f_{-1}(j)}{j}\sum_{l=2}^{\infty} lc(l,j) - af_{-1}(k)\sum_{l=2}^{\infty} lc(l,k) =$$

$$= a\sum_{j=1}^{k} j^{a+1}\frac{\Gamma(2a+1)}{\Gamma(a)j^{a+1}}(1+O(1/j)) + \sum_{j=1}^{k-1} O\left(\frac{(\ln j)^{\lceil a\rceil}}{j}\right) + O\left((\ln k)^{\lceil a\rceil}\right) =$$

$$= ak\frac{\Gamma(2a+1)}{\Gamma(a)} + \sum_{j=1}^{k-1} O\left(\frac{(\ln j)^{\lceil a\rceil}}{j}\right) + O\left((\ln k)^{\lceil a\rceil}\right) = ak\frac{\Gamma(2a+1)}{\Gamma(a)}\left(1+O\left(\frac{(\ln k)^{\lceil a+1\rceil}}{k}\right)\right).$$

Here we used (6) and Lemma 8. We obtain

$$x_k = \frac{a\Gamma(2a+1)}{\Gamma(a)k^{a+1}}\left(1+O\left(\frac{(\ln k)^{\lceil a+1\rceil}}{k}\right)\right)$$

and

$$\sum_{l=1}^{\infty} c(l,k) = c(1,k) + x_k = \frac{(a+1)\Gamma(2a+1)}{\Gamma(a)k^{a+1}}\left(1+O\left(\frac{(\ln k)^{\lceil a+1\rceil}}{k}\right)\right).$$

4.2.3 Estimation of $\mathsf{E}Y_n(k)$

Note that

$$\sum_{l\geqslant 1}\sum_{j\geqslant k}\mathsf{E}N_{i+1}(l,j) = \sum_{l\geqslant 1}\sum_{j\geqslant k}\mathsf{E}N_i(l,j) + \sum_{l\geqslant 1}\frac{(al+k-1)\mathsf{E}N_i(l,k-1)}{(a+1)i+a} + \sum_{j\geqslant k}\frac{(j-1+a)\,M_i^1(j)}{(a+1)i+a}.$$

Therefore we obtain

$$\sum_{l\geqslant 1}\sum_{j\geqslant k}\mathsf{E}N_n(l,j) = \sum_{i=1}^{n-1}\sum_{l\geqslant 1}\frac{(al+k-1)\mathsf{E}N_i(l,k-1)}{(a+1)i+a} + \sum_{i=1}^{n-1}\sum_{j\geqslant k}\frac{(j-1+a)\,M_i^1(j)}{(a+1)i+a}.$$

Let us estimate the sum

$$\sum_{i=1}^{n-1}\sum_{j\geqslant k}\frac{(j-1+a)\,M_i^1(j)}{(a+1)i+a}.$$

First we compute
$$F_t(k) = \sum_{j \geqslant k} (j-1+a)\, M_t^1(j).$$
Let us prove by induction on k that
$$F_n(k) = \frac{(a+1)\Gamma(2a+1)\Gamma(k+a)}{\Gamma(a+1)\Gamma(k+2a)} n \left(1 + \theta\left(\frac{C(k-1)^{1+a}}{n}\right)\right)$$
with some constant C. For $k=1$ and $k=2$ we have
$$F_n(1) = \sum_{j \geqslant 1} (j-1+a)\, M_n^1(j) = n(1+a),$$
$$F_n(2) = F_n(1) - a\, M_n^1(1) = n(1+a) - an\frac{(1+a)}{(2a+1)} + O\left(1\right) = \frac{(1+a)^2 n}{2a+1}\left(1 + O\left(1/n\right)\right).$$
For $k \geqslant 3$ we have
$$M_{i+1}^1(j) = M_i^1(j)\left(1 - \frac{j-1+a}{(a+1)i+a}\right) + M_i^1(j-1)\frac{j-2+a}{(a+1)i+a}.$$
We multiply this equality by $(j-1+a)$ and sum over all $j \geqslant k$:
$$F_{i+1}(k) = \sum_{j \geqslant k} (j-1+a)\, M_{i+1}^1(j) =$$
$$= \sum_{j \geqslant k} (j-1+a) M_i^1(j) - \sum_{j \geqslant k} M_i^1(j)\frac{(j-1+a)(j-1+a)}{(a+1)i+a} + \sum_{j \geqslant k-1} M_i^1(j)\frac{(j+a)(j-1+a)}{(a+1)i+a} =$$
$$= F_i(k) + \sum_{j \geqslant k} M_i^1(j)\frac{(j-1+a)}{(a+1)i+a} + M_i^1(k-1)\frac{(k-1+a)(k-2+a)}{(a+1)i+a} =$$
$$= F_i(k)\left(1 + \frac{1}{(a+1)i+a}\right) + \left(F_i(k-1) - F_i(k)\right)\frac{(k-1+a)}{(a+1)i+a} =$$
$$= F_i(k)\left(1 - \frac{k-2+a}{(a+1)i+a}\right) + F_i(k-1)\frac{(k-1+a)}{(a+1)i+a}.$$
Note that for $i+1 < k-1$ we have $F_{i+1}(k) = 0$. Consider $i+1 \geqslant k-1$. Using the inductive assumption we get
$$F_{i+1}(k) = \frac{(a+1)\Gamma(2a+1)\Gamma(k+a)}{\Gamma(a+1)\Gamma(k+2a)} i \left(1 - \frac{k-2+a}{(a+1)i+a}\right)\left(1 + \theta\left(\frac{C(k-1)^{1+a}}{i}\right)\right) +$$
$$+ \frac{(a+1)\Gamma(2a+1)\Gamma(k-1+a) i}{\Gamma(a+1)\Gamma(k-1+2a)} \frac{(k-1+a)}{(a+1)i+a}\left(1 + \theta\left(\frac{C(k-2)^{1+a}}{i}\right)\right) =$$
$$= \frac{(a+1)\Gamma(2a+1)\Gamma(k+a)}{\Gamma(a+1)\Gamma(k+2a)}\left(i + 1 - \frac{a}{(a+1)i+a} + i\left(1 - \frac{k-2+a}{(a+1)i+a}\right)\theta\left(\frac{C(k-1)^{1+a}}{i}\right) +\right.$$

$$+\frac{(k-1+2a)i}{(a+1)i+a}\theta\left(\frac{C(k-2)^{1+a}}{i}\right)\Bigg).$$

And we need to show that for some constant C

$$\frac{a}{(a+1)i+a}+\frac{(k-1+2a)}{(a+1)i+a}C(k-2)^{1+a}\leqslant\frac{k-2+a}{(a+1)i+a}C(k-1)^{1+a}.$$

This inequality holds for sufficiently large C.

We have

$$\sum_{i=1}^{n-1}\frac{F_i(k)}{(a+1)i+a}=\sum_{i=1}^{n-1}\frac{\Gamma(2a+1)\Gamma(k+a)}{\Gamma(a+1)\Gamma(k+2a)}\left(1+O\left(\frac{k^{1+a}}{i}\right)\right)=$$

$$=\frac{\Gamma(2a+1)\Gamma(k+a)}{\Gamma(a+1)\Gamma(k+2a)}n\left(1+O\left(\frac{k^{1+a}}{n}\right)\right).$$

Let us estimate the sum

$$\sum_{i=1}^{n-1}\sum_{l\geqslant1}\frac{(al+k-1)\mathsf{E}N_i(l,k-1)}{(a+1)i+a}.$$

We start with the sum

$$\sum_{l\geqslant1}(al+k-1)\mathsf{E}N_i(l,k-1).$$

It is easy to see that

$$\mathsf{E}N_i(l,k)=O(c(l,k)i).$$

To verify this, one can follow the proof of Theorem 4 and make sure that it works for the inequality

$$\mathsf{E}N_i(l,k)<\tilde{C}c(l,k)((a+1)i+a)$$

with some constant $\tilde{C}$ – note that the analog of Lemma 1 is also needed.

Therefore

$$\sum_{l\geqslant1}(al-1)\mathsf{E}N_i(l,k-1)=O\left(\sum_{l\geqslant1}(al-1)c(l,k-1)i\right)=O\left(\frac{(\ln k)^{\lceil a\rceil}i}{k^{a+1}}\right).$$

Using (5) we obtain

$$\sum_{l\geqslant1}k\mathsf{E}N_i(l,k-1)=\sum_{l\geqslant1}kc(l,k-1)i\left(1+O\left((l+k)^{1+a}/i\right)\right)=$$

$$=\frac{(a+1)\Gamma(2a+1)}{\Gamma(a)k^a}i\left(1+O\left(\frac{(\ln k)^{\lceil a+1\rceil}}{k}\right)+O\left(\frac{k^{1+a}}{i}\right)\right).$$

Here we used the following estimate:

$$\sum_{l\geqslant1}kc(l,k-1)(l+k)^{1+a}=O\left(\sum_{l=1}^{k}k^{2+a}c(l,k-1)+\sum_{l\geqslant k}kc(l,k-1)l^{1+a}\right)=O(k)+O(1)=O(k).$$

So we have

$$\sum_{i=1}^{n-1}\sum_{l\geqslant 1}\frac{(al+k-1)\mathsf{E}N_i(l,k-1)}{(a+1)i+a}=\frac{\Gamma(2a+1)}{\Gamma(a)k^a}n\left(1+O\left(\frac{(\ln k)^{\lceil a+1\rceil}}{k}\right)+O\left(\frac{k^{1+a}}{n}\right)\right).$$

Hence

$$\sum_{l\geqslant 1}\sum_{j\geqslant k}\mathsf{E}N_n(l,j)=\frac{a\Gamma(2a+1)}{\Gamma(a+1)k^a}n\left(1+O\left(\frac{(\ln k)^{\lceil a+1\rceil}}{k}\right)\right)+\frac{\Gamma(2a+1)\Gamma(k+a)}{\Gamma(a+1)\Gamma(k+2a)}n\left(1+O\left(\frac{k^{1+a}}{n}\right)\right)=$$

$$=\frac{(a+1)\Gamma(2a+1)}{\Gamma(a+1)k^a}n\left(1+O\left(\frac{(\ln k)^{\lceil a+1\rceil}}{k}\right)+O\left(\frac{k^{1+a}}{n}\right)\right).$$

Consider vertices with loops. For $k=0$, we have

$$\sum_{l\geqslant 1}\sum_{j\geqslant 0}\mathsf{E}P_n(l,j)=\sum_{i=1}^{n}\frac{a}{(1+a)i-1}=O(\ln n).$$

For $k\geqslant 2$, we have

$$\sum_{l\geqslant 1}\sum_{j\geqslant k}\mathsf{E}P_{i+1}(l,j)=\sum_{l\geqslant 1}\sum_{j\geqslant k}\mathsf{E}P_i(l,j)+\sum_{l\geqslant 1}\frac{(al+k-2a-1)\mathsf{E}P_i(l,k-1)}{(a+1)i+a}.$$

Therefore, we obtain

$$\sum_{l\geqslant 1}\sum_{j\geqslant k}\mathsf{E}P_n(l,j)=\sum_{i=1}^{n-1}\sum_{l\geqslant 1}\frac{(al+k-2a-1)\mathsf{E}P_i(l,k-1)}{(a+1)i+a}\leqslant\sum_{i=1}^{n-1}\sum_{l\geqslant 1}\frac{(al+k-2a-1)p(l,k-1)}{(a+1)i}.$$

From the recurrent relation for $p(l,k)$ it follows that

$$p(l,k)=O\left(\frac{1}{l^2}\right),$$

and

$$p(l,k)=O\left(\frac{k}{l^3}\right).$$

To obtain the second estimate consider $q(l,k)=p(l,k)/k$. For $k\geqslant 1$ we have

$$q(l,k)(l+al+k-1-a)=q(l,k-1)\frac{(k-1)(al+k-2a-1)}{k}+q(l-1,k)(l-2+a),$$

$$q(l,k)(l+al+k-1-a)-q(l,k-1)\left(al+k-2a-2-\frac{al-2a-1}{k}\right)=q(l-1,k)(l-2+a).$$

Thus, $q(l,k)=O(q(l))$, where

$$q(l)(l+a+1+(al-2a-1))=q(l-1)(l-2+a).$$

From this equality it follows that $q(l) = O\left(\frac{1}{l^3}\right)$.

We can estimate the following sum:

$$\sum_{l \geqslant 1} \frac{(al + k - 2a - 1)p(l, k-1)}{(a+1)} = O\left(k\right).$$

Hence

$$\sum_{l \geqslant 1} \sum_{j \geqslant k} \mathsf{E} P_n(l, j) = O(k \ln n).$$

Now we are ready to estimate $\mathsf{E} Y_n(k)$:

$$\mathsf{E} Y_n(k) = \sum_{l \geqslant 1} \sum_{j \geqslant k} \mathsf{E} N_n(l, j) + \sum_{l \geqslant 1} \sum_{j \geqslant k} \mathsf{E} P_n(l, k) =$$

$$= \frac{(a+1)\Gamma(2a+1)}{\Gamma(a+1)k^a} n \left(1 + O\left(\frac{(\ln k)^{\lceil a+1 \rceil}}{k}\right) + O\left(\frac{k^{1+a}}{n}\right)\right) + O(k \ln n) =$$

$$= \frac{(a+1)\Gamma(2a+1)}{\Gamma(a+1)k^a} n \left(1 + O\left(\frac{(\ln k)^{\lceil a+1 \rceil}}{k}\right) + O\left(\frac{k^{1+a}}{n}\right)\right).$$

This concludes the proof of Theorem 2.

4.3 Proof of Lemma 2

It is easy to see that $\mathsf{E} P_n(0, k) = \mathsf{E} P_n(1, k) = 0$. For all $k > 0$ we have $\mathsf{E} P_n(2, k) = 0$. For $k = 0$ we have

$$\mathsf{E} P_n(2, 0) = \sum_{i=1}^{n} \frac{a}{(a+1)i - 1} \prod_{j=i+1}^{n} \frac{(1+a)j - 2 - a}{(1+a)j - 1} = \sum_{i=1}^{n} \frac{a}{(a+1)i - 1} \frac{\Gamma\left(n - \frac{1}{a+1}\right)\Gamma\left(i + \frac{a}{a+1}\right)}{\Gamma\left(n + \frac{a}{a+1}\right)\Gamma\left(i - \frac{1}{a+1}\right)} =$$

$$= \frac{1}{n}(1 + O(1/n)) \sum_{i=1}^{n} \frac{ai}{(a+1)i - 1}(1 + O(1/i)) = O(1).$$

The rest of the proof is by induction. Consider $l \geqslant 3$, $k \geqslant 1$. Assume that we already proved that $\mathsf{E} P_n(i, j) \leqslant p(i, j)$ for all i and j, such that $i < l$, $j \leqslant k$ or $i \leqslant l$, $j < k$. We use the following equality

$$\mathsf{E} P_{i+1}(l, k) = \mathsf{E} P_i(l, k)\left(1 - \frac{l(a+1) + k - a - 1}{(a+1)i + a}\right) + \mathsf{E} P_i(l, k-1)\frac{al + k - 2a - 1}{(a+1)i + a} + \\ + \mathsf{E} P_i(l-1, k)\frac{l - 2 + a}{(a+1)i + a}. \quad (7)$$

Note that if we have at least one vertex with a loop, with first degree l and second degree k in the graph $H^i_{a,1}$, then we have at least $l + k - 1$ edges in this graph. Therefore $\mathsf{E} P_i(l, k) = 0$ if $i < l + k - 1$. Consider the case $i = l + k - 1$. Using (7), we get (for $k \geqslant 1$)

$$\mathsf{E} P_{l+k-1}(l, k) = \frac{(l - 2 + a)\,\mathsf{E} P_{l+k-2}(l-1, k)}{(a+1)(l+k-2) + a} + \frac{(al + k - 2a - 1)\mathsf{E} P_{l+k-2}(l, k-1)}{(a+1)(l+k-2) + a} \leqslant$$

$$\leqslant \frac{(l-2+a)p(l-1,k)}{(a+1)(l+k-2)+a} + \frac{(al+k-2a-1)p(l,k-1)}{(a+1)(l+k-2)+a} =$$

$$= \frac{(l+al+k-1-a)p(l,k)}{(a+1)(l+k-2)+a} \leqslant p(l,k).$$

The last inequality holds for $k \geqslant 1/a$. Consider the case $k < 1/a$. As in proof of Theorem 4, at first we estimate $p(l,k)$:

$$p(l,k) = \Omega\left(\frac{l^{k+\frac{a^2}{a+1}}}{(1+a)^l}\right).$$

For $k = 0$ we have

$$p(l,0) = p(l-1,0)\frac{l-2+a}{(1+a)(l-1-\frac{1}{a+1})}.$$

Therefore,

$$p(l,0) = \Omega\left(\frac{l^{\frac{a^2}{a+1}}}{(1+a)^l}\right).$$

For $k \geqslant 1$ we have

$$p(l,k) = p(l,k-1)\frac{al+k-2a-1}{l(1+a)+k-1-a} + p(l-1,k)\frac{l-2+a}{l(1+a)+k-1-a}.$$

Again, it is sufficient to prove that there exists a positive function $f(k)$ such that for big l

$$f(k)(l(1+a)+k-1-a)l^{k+\frac{a^2}{a+1}} \leqslant f(k-1)(al+k-2a-1)l^{k+\frac{a^2}{a+1}-1} + f(k)(l-2+a)(a+1)(l-1)^{k+\frac{a^2}{a+1}},$$

$$f(k)(l(1+a)+k-1-a)\left(l^{k+\frac{a^2}{a+1}} - (l-1)^{k+\frac{a^2}{a+1}}\right) + f(k)(k-a^2+1)(l-1)^{k+\frac{a^2}{a+1}} \leqslant$$

$$\leqslant f(k-1)(al+k-2a-1)l^{k+\frac{a^2}{a+1}-1}.$$

The last inequality holds for some function $f(k)$.

We want to prove that

$$\mathsf{E}P_{l+k-1}(l,k) = O\left(\frac{l^{k+\frac{a^2}{a+1}}}{(1+a)^l}\right).$$

There are l^k possible graphs on $l+k-1$ vertices with some vertex of first degree l, second degree k, and without a loop. And this vertex is exactly the vertex 1. The probability of this vertex to be a vertex with first degree l and second degree k equals

$$O\left(\frac{l^k((a+1)\dots(a+l-2))}{(a+2)\dots((l-1)(a+1)-1)}\right) = O\left(\frac{l^{k+\frac{a^2}{a+1}}}{(a+1)^l}\right).$$

This concludes the case $i = l+k-1$.

If $i \geqslant l+k-1$, then

$$\mathsf{E}P_{i+1}(l,k) = \mathsf{E}P_i(l,k)\left(1 - \frac{l(a+1)+k-a-1}{(a+1)i+a}\right) +$$

$$+\mathsf{E}P_i(l,k-1)\frac{al+k-2a-1}{(a+1)i+a} + \mathsf{E}P_i(l-1,k)\frac{l-2+a}{(a+1)i+a}.$$

Using the recurrent relation for $p(l,k)$ and induction on i it is easy to prove that $\mathsf{E}P_n(l,k) \leqslant p(l,k)$. This concludes the proof of Lemma 2.

4.4 Proof of Theorem 3

We estimate the expectation of $X_n(k)$ as follows:

$$\mathsf{E}X_n(k) = \sum_{l=1}^{\infty}\mathsf{E}N_n(l,k) + \sum_{l=1}^{\infty}\mathsf{E}P_n(l,k) = \sum_{l=1}^{\infty}c(l,k)n + O\left(\sum_{l=1}^{\infty}c(l,k)(l+k)^{1+a}\right) + O\left(\sum_{l=1}^{\infty}p(l,k)\right) =$$

$$= \frac{(a+1)\Gamma(2a+1)n}{\Gamma(a)k^{a+1}}\left(1 + O\left(\frac{(\ln k)^{\lceil a+1\rceil}}{k}\right)\right) + O(1) + O(1) =$$

$$= \frac{(a+1)\Gamma(2a+1)n}{\Gamma(a)k^{a+1}}\left(1 + O\left(\frac{(\ln k)^{\lceil a+1\rceil}}{k}\right) + O\left(\frac{k^{1+a}}{n}\right)\right).$$

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